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Series Solutions, Factorials, And The Gamma Function

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Series Solutions

Suppose we want to find a power series solution of the differential equation

$$y'' + 3xy' + y = 0$$

about x = 0. Since zero is an ordinary point, we know that a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

can be obtained. By substituting the solution form into the differential equation, the recursion relation

$$a_{n+2} = \frac{-(3n+1)a_n}{(n+2)(n+1)}$$

can be obtained. Using this recursion relation, we can determine the values of the coefficients of the series solution, in terms of a_0 and a_1. Some of these values are

$$a_2 = \frac{-a_0}{2 \cdot 1}$$

$$a_4 = \frac{-7a_2}{4 \cdot 3} = \frac{1 \cdot 7a_0}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$a_6 = \frac{-13a_4}{6 \cdot 5} = \frac{-1 \cdot 7 \cdot 13a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

We have purposefully chosen to examine only the coefficients with even subscripts, as only they depend on a_0. An analysis similar to the foregoing can also be done with the coefficients having odd subscripts.

At this point, if we desire a general nth term, we must examine our coefficients for a pattern. It appears that the coefficients listed above alternate in sign, have factorial expressions in the denominator, and contain a product of the terms of a finite arithmetic sequence in the numerator. From college algebra, we know that arithmetic sequences can be described by linear functions. Therefore, we can make the hypothesis that the general (2n)th term is

$$a_{2n} = \frac{(-1)^n 1 \cdot 7 \cdot 13 \cdot \cdots \cdot (6n-5)\ a_0}{(2n)!}$$

In order to reach this conclusion, notice that we expressed the recursion relation in <u>completely factored form</u>. Neither the linear factors of the recursion relation nor the numerical quantities which they produced were multiplied out. To do so would obscure the mathematical structure of both the recursion relation and the formula for the general term.

It is possible to prove that the formula for the general term we produced is, in fact, valid. A proof by mathematical induction is needed. First, we shall verify that the formula is valid for $n = 1$. Substituting 1 for n in the formula, we obtain

$$a_{2(1)} = \frac{(-1)^1 \, 1a_0}{(2 \cdot 1)!} = \frac{-a_0}{2}$$

which is identical to the value obtained earlier using the recursion relation. Then we must show that the formula for a_{2n} implies the formula for $a_{2(n+1)}$. To do this, we compare the value of a_{2n+2} obtained by the recursion relation,

$$\begin{aligned} a_{2n+2} &= \frac{-(3(2n)+1)a_{2n}}{(2n+2)(2n+1)} \\ &= \frac{-(6n+1)}{(2n+2)(2n+1)} \frac{(-1)^n \, 1 \cdot 7 \cdot 13 \cdot \cdots \cdot (6n-5) \; a_0}{(2n)!} \\ &= \frac{(-1)^{n+1} \, 1 \cdot 7 \cdot 13 \cdot \cdots \cdot (6n+1) \; a_0}{(2n+2)!} \end{aligned}$$

with the value obtained by our formula

$$\begin{aligned} a_{2n+2} &= \frac{(-1)^{n+1} \, 1 \cdot 7 \cdot 13 \cdot \cdots \cdot (6(n+1)-5) \; a_0}{(2(n+1))!} \\ &= \frac{(-1)^{n+1} \, 1 \cdot 7 \cdot 13 \cdot \cdots \cdot (6n+1) \; a_0}{(2n+2)!} \end{aligned}$$

These two expressions are the same. We have shown that the formula is correct for any term having an even subscript whenever it was also correct for the term with the previous even subscript. Since the formula was correct for a_2, it will be correct for all a_n with even subscripts larger than 2.

Factorials

Looking back, we notice that we easily recognized the pattern

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 6!$$

but we did not have a similar insight to the pattern

$$1 \cdot 7 \cdot 13 \cdot \cdots$$

except to say that the factors had the form $(6n - 5)$. The purpose of this unit is to show how any such pattern (where the factors are the terms of a finite arithmetic sequence) can be written in a more concise form.

The easiest such pattern to write concisely is the product of a finite number of consecutive integers. We simply recognize

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 6!$$

or

$$1 \cdot 2 \cdot 3 \cdot \cdots \cdot n = n!$$

If the sequence of integers in the product does not begin with one, we can make a minor adjustment. For example,

$$5 \cdot 6 \cdot 7 \cdot 8 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{8!}{4!}$$

or

$$m\,(m+1)\,(m+2) \cdot \cdots \cdot (m+n) = \frac{(m+n)!}{(m-1)!}$$

Consecutive multiples of any integer can also be handled with factorials. For example,

$$\begin{aligned} 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n) &= (2 \cdot 1)(2 \cdot 2)(2 \cdot 3) \cdot \cdots \cdot (2 \cdot n) \\ &= 2^n(1 \cdot 2 \cdot 3 \cdot \cdots \cdot n) \\ &= 2^n\, n! \end{aligned}$$

and

$$\begin{aligned} 7 \cdot 14 \cdot 21 \cdot \cdots \cdot (7n) &= (7 \cdot 1)(7 \cdot 2)(7 \cdot 3) \cdot \cdots \cdot (7 \cdot n) \\ &= 7^n(1 \cdot 2 \cdot 3 \cdot \cdots \cdot n) \\ &= 7^n\, n! \end{aligned}$$

A few other cases can be handled in a similar fashion, but a more general procedure is needed. In order to have such a procedure, we shall introduce the Gamma Function.

The Gamma Function

The Gamma Function is defined by an improper definite integral, specifically

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

In order to better understand how the definition works, we shall find the value of this function at $x = 1$.

$$\begin{aligned} \Gamma(1) &= \int_0^\infty t^{1-1}e^{-t}\,dt = \int_0^\infty e^{-t}\,dt \\ &= \lim_{c\to\infty}\int_0^c e^{-t}\,dt = \lim_{c\to\infty}[-e^{-t}]_0^c \\ &= \lim_{c\to\infty}[-e^{-c} + e^0] = 0 + 1 = 1 \end{aligned}$$

When defining a new function, one major concern we should have is the domain of the function. That is, where is the definition valid? Since we have defined the Gamma Function in terms of an improper integral, our question is basically equivalent to finding the interval of convergence.

Since, for any real value of x, the integrand is a continuous function of t on the interval $(0, \infty)$, we can write

$$\Gamma(x) = \int_0^1 t^{x-1}e^{-t}\,dt + \int_1^\infty t^{x-1}e^{-t}\,dt$$

Depending on the value of x, either one or both of these integrals are improper, and we are concerned about their convergence. Now, the integral $\int_1^{\infty} t^{x-1}e^{-t}\,dt$ and the sum $\sum_{n=1}^{\infty} n^{x-1}e^{-n}$ either both converge or both diverge, according to the Integral Test for infinite series. Using the Ratio Test for infinite series, we can analyze the infinite series to find

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^{x-1}e^{-(n+1)}}{n^{x-1}e^{-n}}\right| \\ &= \lim_{n\to\infty}\frac{1}{e}\left(1+\frac{1}{n}\right)^{x-1} \\ &= \frac{1}{e} < 1\end{aligned}$$

Since the limit is less than one, the Ratio Test determines that the infinite series converges, and therefore (by the Integral Test), so does the integral.

Now let us consider the other integral. When x is greater than or equal to 1, the integrand is continuous on [0,1], and therefore integrable. When x is positive but less than one, the integrand is unbounded at $t = 0$, and therefore we must use limits. Integrating by parts, we find

$$\begin{aligned}\int_0^1 t^{x-1}e^{-t}\,dt &= \lim_{c\to 0}\int_c^1 t^{x-1}e^{-t}\,dt \\ &= \lim_{c\to 0}\left[\frac{1}{x}t^{x}e^{-t}\right]_c^1 + \frac{1}{x}\lim_{c\to 0}\int_c^1 t^{x}e^{-t}\,dt\end{aligned}$$

For any fixed positive value of x, the expression $t^{x}e^{-t}$ is continuous on the t-interval [0,1], and therefore both the integral and the argument of the limit exist. By the discussion in this and the preceding paragraph, we see that the Gamma Function is well defined for all positive values of x. If the integration by parts shown in this paragraph had been carried out a few more times, we could also establish the result for all real numbers, except for the negative integers and zero.

The Gamma Function has a property similar to a property of factorials. It is called the Recursion Formula, and is given by

$$\Gamma(x+1) = x\,\Gamma(x)$$

We shall use the definition of the Gamma Function and integration by parts to provide the proof. Given that x is <u>not</u> a negative integer or zero, then

$$\begin{aligned}\Gamma(x+1) &= \int_0^{\infty} t^{x}\,e^{-t}\,dt \\ &= \left[-t^{x}e^{-t}\right]_0^{\infty} + \int_0^{\infty} xt^{x-1}\,e^{-t}\,dt \\ &= \lim_{a\to\infty}\left[-a^{x}e^{-a}\right] - \lim_{b\to 0}\left[-b^{x}e^{-b}\right] + x\int_0^{\infty} t^{x-1}\,e^{-t}\,dt\end{aligned}$$

The first limit is zero. (If x is positive, L'Hopital's Rule can be used.) The second limit is also zero. (If x is negative, L'Hopital's Rule can be used.) The term containing the integral is equal to the right hand side of the Recursion Formula.

Earlier, we found that $\Gamma(1) = 1$. With the Recursion Formula, we can find many other values of the Gamma Function, especially for positive integer arguments.

$$\begin{aligned} \Gamma(2) &= 1 \cdot \Gamma(1) = 1 \\ \Gamma(3) &= 2 \cdot \Gamma(2) = 2 \\ \Gamma(4) &= 3 \cdot \Gamma(3) = 6 \\ &\dots \\ \Gamma(n) &= (n-1)\,\Gamma(n-1) = (n-1)! \end{aligned}$$

Most other values of the Gamma Function are quite difficult to find. For example, to find the value of $\Gamma\left(\frac{1}{2}\right)$, we begin by using the substitution $t = u^2$.

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt \\ &= \int_0^\infty \frac{1}{u} e^{-u^2} (2u\, du) \\ &= \int_0^\infty 2e^{-u^2}\, du \end{aligned}$$

The last integrand does not have an antiderivative expressible in terms of elementary functions, but we can evaluate this definite integral nevertheless. To do so, we shall consider the square of $\Gamma\left(\frac{1}{2}\right)$, treat the integrand as a function of two variables defined in the first quadrant of the plane, and then change to polar coordinates.

$$\begin{aligned} \left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= \left[\int_0^\infty 2e^{-u^2}\, du\right]^2 \\ &= \left[\int_0^\infty 2e^{-x^2}\, dx\right]\left[\int_0^\infty 2e^{-y^2}\, dy\right] \\ &= \int_0^\infty\int_0^\infty 4e^{-(x^2+y^2)}\, dx\, dy \\ &= \int_0^{\frac{\pi}{2}}\int_0^\infty 4re^{-r^2}\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[-2e^{-r^2}\right]_0^\infty d\theta \\ &= \int_0^{\frac{\pi}{2}} 2\, d\theta = [2\theta]_0^{\frac{\pi}{2}} = \pi \end{aligned}$$

Therefore

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Furthermore, using the Recursion Formula, we also find

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$
$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}$$
$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

and if we rewrite the Recursion Formula as

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

we can also find

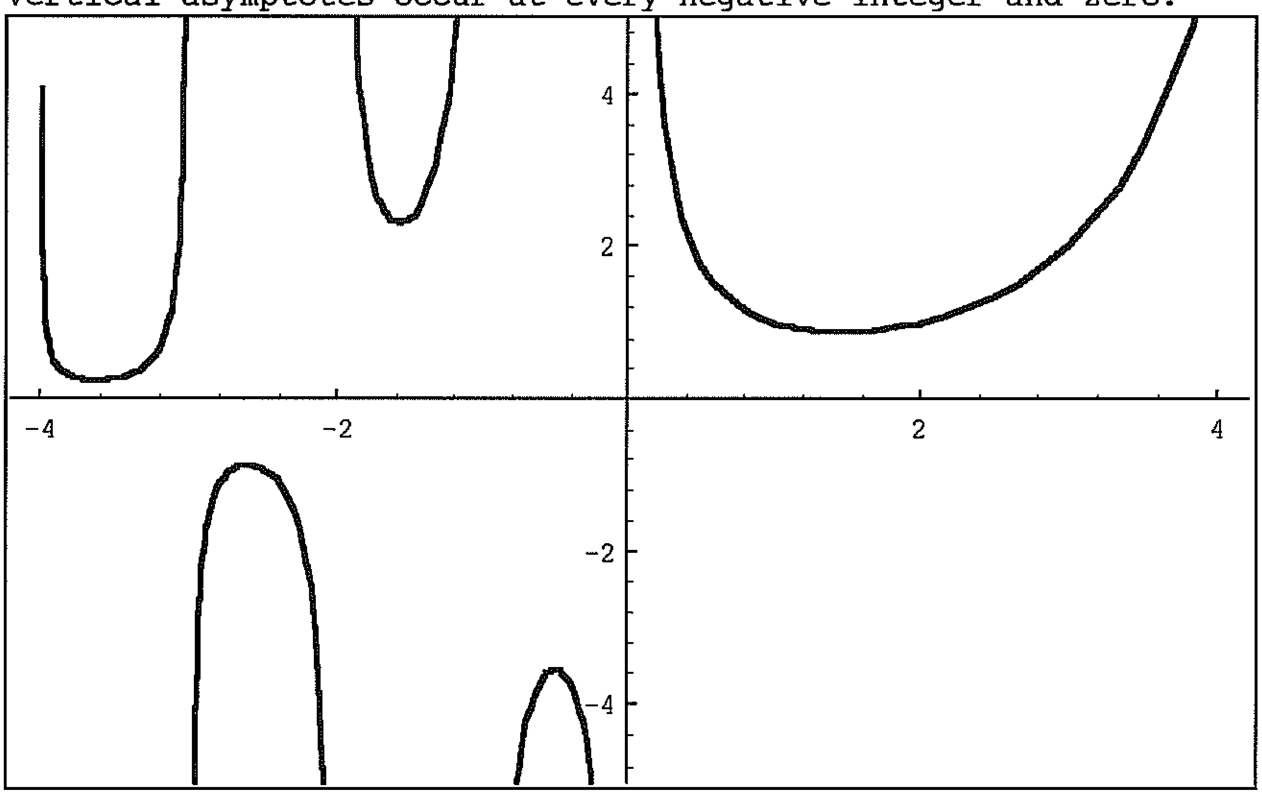
$$\Gamma\left(-\frac{1}{2}\right) = -\frac{2}{1}\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$
$$\Gamma\left(-\frac{3}{2}\right) = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

A third way in which the Recursion Formula can be written is

$$x = \frac{\Gamma(x+1)}{\Gamma(x)}$$

This form will be useful for us later, since it allows us to write any number in the domain of the Gamma Function as the quotient of two values of that function.

Here is a graph of some of the values of the Gamma Function. Vertical asymptotes occur at every negative integer and zero.



Product of a Finite Arithmetic Sequence

Our original reason for examining the Gamma Function was the claim that it would allow us to more concisely write the product of any finite arithmetic sequence of numbers, which, in turn, can be used to write series solution results more concisely. Now we shall see how this can be done.

Consider the product

$$3 \cdot 7 \cdot 11 \cdot \cdots \cdot 43$$

The sequence of numbers has a common difference of four. Multiplying and dividing each factor by 4, and collecting all of the fours from the numerator in a separate factor, we obtain

$$4^{11}\left(\frac{3}{4} \cdot \frac{7}{4} \cdot \frac{11}{4} \cdot \cdots \cdot \frac{43}{4}\right)$$

This sequence of fractions has a common difference of one. Rewriting each of the fractions as the quotient of two Gamma Functions, we get

$$4^{11}\,\frac{\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \cdot \frac{\Gamma\left(\frac{11}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} \cdot \frac{\Gamma\left(\frac{15}{4}\right)}{\Gamma\left(\frac{11}{4}\right)} \cdot \cdots \cdot \frac{\Gamma\left(\frac{47}{4}\right)}{\Gamma\left(\frac{43}{4}\right)}$$

After a great deal of cancellation, we find that

$$3 \cdot 7 \cdot 11 \cdot \cdots \cdot 43 = 4^{11}\frac{\Gamma\left(\frac{47}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

Let us consider another example, namely

$$17 \cdot 10 \cdot 3 \cdot (-4) \cdot \cdots \cdot (24 - 7n)$$

We note that the linear factor at the end of the expression gives the first four factors of this product for the values of n from 1 to 4. Therefore, the product contains n factors. We can rewrite it as

$$\begin{aligned} &17 \cdot 10 \cdot 3 \cdot \cdots \cdot (24 - 7n) \\ &= 7^n \cdot \frac{17}{7} \cdot \frac{10}{7} \cdot \frac{3}{7} \cdot \cdots \cdot \left(\frac{24}{7} - n\right) \\ &= 7^n \cdot \frac{\Gamma\left(\frac{24}{7}\right)}{\Gamma\left(\frac{17}{7}\right)} \cdot \frac{\Gamma\left(\frac{17}{7}\right)}{\Gamma\left(\frac{10}{7}\right)} \cdot \frac{\Gamma\left(\frac{10}{7}\right)}{\Gamma\left(\frac{3}{7}\right)} \cdot \cdots \cdot \frac{\Gamma\left(\frac{31}{7} - n\right)}{\Gamma\left(\frac{24}{7} - n\right)} \\ &= 7^n \cdot \frac{\Gamma\left(\frac{24}{7}\right)}{\Gamma\left(\frac{24}{7} - n\right)} \end{aligned}$$

Rewriting the Series Solution Result

Let us now go back to our original differential equation

$$y'' + 3xy' + y = 0$$

Previously, we had found that the recursion relation was

$$a_{n+2} = \frac{-(3n+1)a_n}{(n+2)(n+1)}$$

This recursion relation produces the formulas

$$a_{2n} = \frac{(-1)^n\, 1 \cdot 7 \cdot 13 \cdot \cdots \cdot (6n-5)\; a_0}{(2n)!}$$

and

$$a_{2n+1} = \frac{(-1)^n\, 4 \cdot 10 \cdot 16 \cdot \cdots \cdot (6n-2)\; a_1}{(2n+1)!} = \frac{(-2)^n\, 2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)\; a_1}{(2n+1)!}$$

The first of these two formulas we had found earlier. The second formula is obtained in a similar fashion. Using the Gamma Function, we can rewrite

$$1 \cdot 7 \cdot 13 \cdot \cdots \cdot (6n-5) = 6^n \frac{\Gamma\left(n + \frac{1}{6}\right)}{\Gamma\left(\frac{1}{6}\right)}$$

and

$$2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1) = 3^n \frac{\Gamma\left(n + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}$$

Therefore, our formulas become

$$a_{2n} = \frac{(-6)^n\, \Gamma\left(n + \frac{1}{6}\right) a_0}{\Gamma\left(\frac{1}{6}\right) (2n)!}$$

and

$$a_{2n+1} = \frac{(-6)^n\, \Gamma\left(n + \frac{2}{3}\right) a_1}{\Gamma\left(\frac{2}{3}\right) (2n+1)!}$$

The solution of the differential equation can then be written as

$$y = \frac{a_0}{\Gamma\left(\frac{1}{6}\right)} \sum_{n=0}^{\infty} \frac{(-6)^n\, \Gamma\left(n + \frac{1}{6}\right)}{(2n)!} x^{2n} + \frac{a_1}{\Gamma\left(\frac{2}{3}\right)} \sum_{n=0}^{\infty} \frac{(-6)^n\, \Gamma\left(n + \frac{2}{3}\right)}{(2n+1)!} x^{2n+1}$$

For practice, you should rewrite, using the Gamma Function, the answers you found for the exercises in the textbook.